Non-monotone Bifurcation on Quadratic Rational Families

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941 巻 1996 年 30-35

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5.1 Moduli space $\mathcal{M}_2(\mathbf{R})$ of real quadratic rational maps

$\mathrm{Rat}_2(\mathbf{R})$ is the space of all real quadratic rational maps $f : \mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$,

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_0 x^2 + a_1 x + a_2}{b_0 x^2 + b_1 x + b_2}.$$

Definition 1 $\mathcal{M}_2(\mathbf{R}) = \mathrm{Rat}_2(\mathbf{R})/\mathrm{PSL}_2(\mathbf{R})$ is called the **moduli space** of holomorphic conjugacy class $\langle f \rangle$ of real quadratic rational maps f.

Remark 1 The definitions of moduli space $\mathcal{M}_2(\mathbf{C})$ for the complex quadratic maps, $\mathrm{Rat}_2(\mathbf{C})$, is identify with $\mathrm{Rat}_2(\mathbf{C})/\mathrm{PSL}_2(\mathbf{C})$.

For each $f \in \mathrm{Rat}_2$, let z_1, z_2, z_3 be fixed points of f, μ_i the multiplier of z_i $(1 \leq i \leq 3)$; $\mu_i = f'(z_i)$. Now consider elementary symmetric functions of three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

Milnor introduces coordinates of $\mathcal{M}_2(\mathbf{C})$ as follows [Mil92].

Lemma 1 (lemma 3.1 of [Mil92]) *These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that*

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0, \tag{1}$$

or in other words

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$, with coordinates σ_1 and σ_2.

Here after we treat only the real case. σ_i $(1 \leq i \leq 3)$ are all real, because three fixed points and multipliers are either all real or one real and a pair of complex conjugate numbers.

Proposition 1 *$\mathcal{M}_2(\mathbf{R})$ is isomorphic to $\mathbf{R}^2$ except on the cubic algebraic curve,*

$$F(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \tag{2}$$

For each (σ_1, σ_2) on this curve, two real representatives $\langle f_1 \rangle, \langle f_2 \rangle$ are determined. These classes correspond under the complex conjugacy $x \mapsto ix$. Although there is the singular curve (2), yet we regard moduli space $\mathcal{M}_2(\mathbf{R})$ as $\mathbf{R}^2$.

Milnor describes the curve (2) implicitly (compare Figure 15 in [Mil92]). Here we can give a defining equation (2) of this cubic curve.

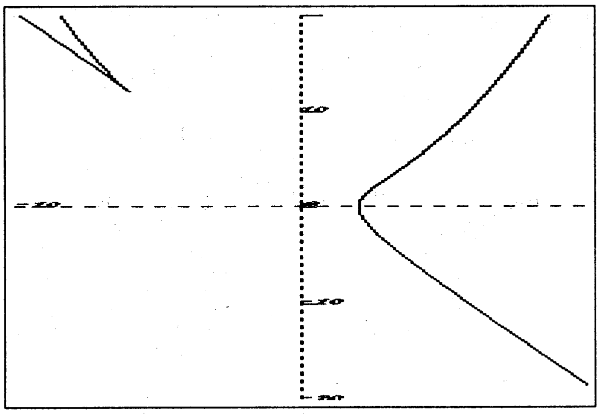

図 1 **Moduli space with** $2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0.$

5.2 A quadratic family with non-monotone bifurcation

M. Bier and T. C. Bountis studied "period-bubbling" bifurcation [BB84]. Their purpose is to demonstrate that monotone bifurcation commonly arise in some of the simplest nonlinear dynamical systems involving the variation of more than one parameter. As a simple example of non-monotone bifurcation, they treat quadratic rational mapping, $x_{t+1} = Q + Ax_t/(x_t^2 + 1)$, $(A, Q > 0)$.

H. E. Nusse and J. A. Yorke gave an example of exponential function family that has non-monotone bifurcation, even though it has negative Schwarzian derivative [NY88]. Their question was arisen of whether having a negative Schwarzian derivative rules out non-monotone bifurcation. They describe in [NY88] that if the above quadratic rational family is written in the following form,

$$\left\{ f_{m,r}(x) = m\frac{rx^2 + x + r}{1 + x^2} \right\},$$

it does not exhibit non-monotone bifurcation as the parameter m is increased. But we can show that this family exhibit non-monotone bifurcation for suitable parameter r.

Since $f_{m,r} \sim f_{m,-r}$, we can regard parameter r as $r \geq 0$.

In general, we obtain next results for a fixed parameter r.

Proposition 2 *On $\mathcal{M}_2(\mathbf{R})$, one parameter family $\{f_{m,r}(x)\}_m$ for a fixed $r \neq \frac{1}{2}, 0$ is characterized as the following irreducible algebraic curve of degree 4,*

$$\begin{aligned} H_r(\sigma_1, \sigma_2) &= -4096r^6 + (-128\sigma_1^2 + 512\sigma_1 + 512\sigma_2 + 1536)r^4 \\ &\quad +(-\sigma_1^4 + 8\sigma_1^3 + (8\sigma_2 + 8)\sigma_1^2 + (-32\sigma_2 - 96)\sigma_1 - 16\sigma_2^2 - 96\sigma_2 - 144)r^2 \\ &\quad -2\sigma_1^3 + (-\sigma_2 + 1)\sigma_1^2 + (8\sigma_2 - 12)\sigma_1 + 4\sigma_2^2 - 12\sigma_2 + 36 = 0. \end{aligned} \tag{3}$$

For $r = \frac{1}{2}$, following irreducible algebraic curve of degree 3.

$$H_{\frac{1}{2}}(\sigma_1, \sigma_2) = -\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0.$$

For $r = 0$,

$$H_0(\sigma_1, \sigma_2) = F(\sigma_1, \sigma_2).$$

Proof. Three fixed points z_1, z_2, z_3 of f are the roots of the equation

$$x^3 - mrx^2 + (1 - m)x - mr = 0.$$

From the relation between coefficients and solutions, following equations hold.

$$\begin{cases} z_1 + z_2 + z_3 = mr \\ z_1z_2 + z_2z_3 + z_3z_1 = 1 - m \\ z_1z_2z_3 = mr \end{cases}$$

Let μ_i $(i = 1, 2, 3)$ be multiplier of each fixed point z_i $(i = 1, 2, 3)$ given by,

$$\mu_i = m\frac{z_i^2 - 1}{(z_i^2 + 1)^2}.$$

By using "Gröbner basis" of Risa/Asir, symbolic and algebraic computation system, we can obtain $\sigma_1(=\mu_1+\mu_2+\mu_3)$ and $\sigma_2(=\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)$ as functions of m and r:

$$\begin{cases} 4m^2r^2-m^2+(\sigma_1+2)m-4=0 \\ -4m^4r^4+(m^4-12m^3-8m^2)r^2+2m^3+(\sigma_2-5)m^2+4m-4=0. \end{cases} \tag{4}$$

Using "Gröbner basis" again, we can remove m from (4), and we have (3).

In the case of $r=\frac{1}{2}$, $-\sigma_1^3-2\sigma_1^2+(4\sigma_2-24)\sigma_1+8\sigma_2-64=0$.

In the case of r equal to 0, algebraic curve of (3) coincides with the curve of (2). ∎

Remark 2 The equation of σ_1 in (4) is obtained from the Program 2. Takeshi Shimoyama(Fujitsu Laboratories) guided me in usage of Risa/Asir and he suggested this program.

Program 2

```
if (vtype(gr)!=3) load("gr")$$
extern Ord$

def moduliS1()
{
        S1=nm(m*((z1^2-1)/(z1^2+1)^2
                +(z2^2-1)/(z2^2+1)^2+(z3^2-1)/(z3^2+1)^2)-s1);
        X=z1+z2+z3-m*r;
        Y=z1*z2+z2*z3+z3*z1-1+m;
        Z=z1*z2*z3-m*r;
        Ord=2;
        G=gr([S1,X,Y,Z],[z1,z2,z3,m,r,s1]);
        for (I=length(G)-1;I>=0;I--){
                E=G[I];
                if (vars(E)==[r,m,s1])
                      break;
        }
        return E;
}
end$
```

To say superfluously, the required equation (3) is obtained from following command of Risa/Asir.

```
gr([4*m^2*r^2-m^2+(s1+2)*m-4,
    -4*m^4*r^4+(m^4-12*m^3-8*m^2)*r^2+2*m^3+(s2-5)*m^2+4*m-4],[m,r]);
```

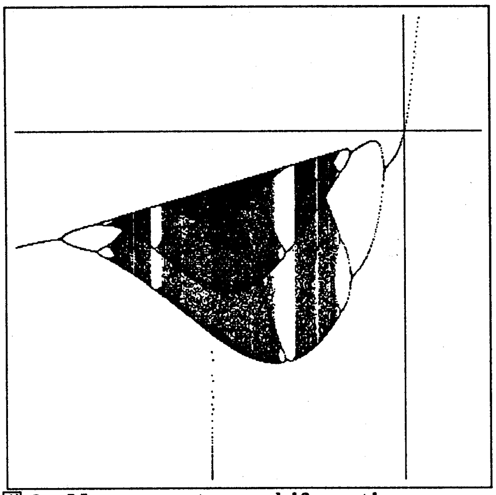

図 2 **Non-monotone bifurcation;** $-25.0 \le m \le 5.0$, $-3.0 \le x \le 1.0$, $r = 0.54$.

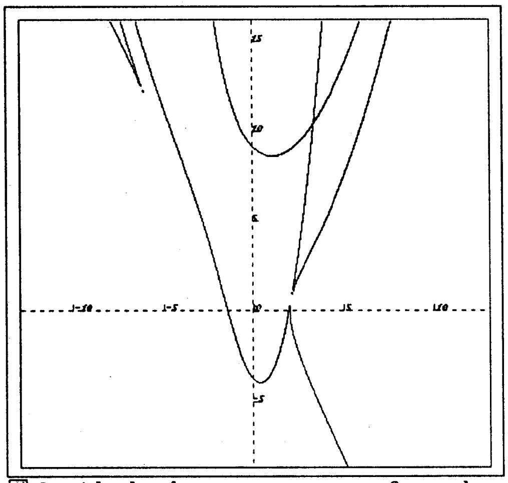

図 3 **Algebraic curve of degree 4 and cubic curve in the moduli space. In the case of** $r = 0.54$.

Example 1 Non-monotone bifurcation can occur at $r = 0.54$, See Figure 2. And its charasteristic curve is Figure 3.

We can analyze the non-monotone bifurcation by overwriting the algebraic curve of degree 4 on the $\mathcal{M}_2(\mathbf{R})$.

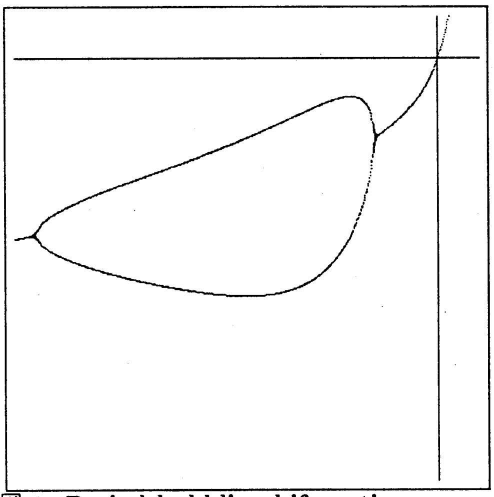

図 4 **Period-bubbling bifurcation:** $-10 \le m \le 1$, $-2 \le x \le 0.2$, **Parameter** $r = 0.58$.

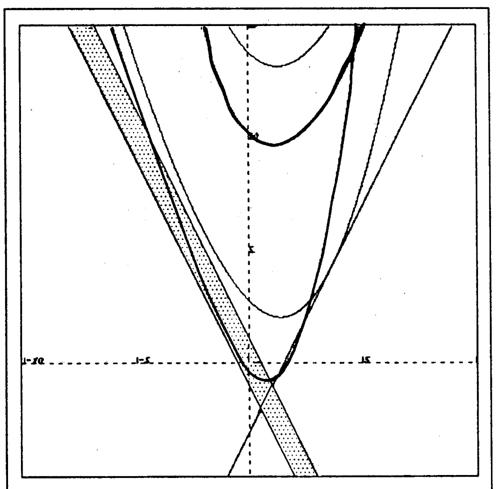

図 5 **Algebraic curves of degree 4 in the "classified" moduli space. Thick curve corresponds with** $r = 0.58$, **thin curve corresponds with** $r = 0.7$.

Example 2 One parameter family $\{f_{m,0.58}\}$ has non-monotone (period-bubbling) bifurcation. See Figure 4.

In Figure 5, the thick line indicates this family, and the gray belt is the region on which each map has attracting period 2 cycle. When algebraic curve of degree 4 through this gray belt, period-doubling bifurcation occurs. In this case, the curve intersects the gray belt (period-doubling occurs) and intersects again the period 1 region (period-halving occurs). Hence period-bubbling bifurcation occurs, as in Figure 4.

Theorem 1 *For a fixed parameter r, there are following three possibilities;*

1. *various bifurcations occur if* $0 < r \le \frac{1}{2}$,
2. *non-monotone bifurcations occur if* $\frac{1}{2} < r < \frac{3\sqrt{3}}{8}$, *or*
3. *any bifurcation can't occur if* $\frac{3\sqrt{3}}{8} \le r$.

Acknowledgements: Takeshi Shimoyama (Fujitsu Laboratories) taught me about Gröbner basis of Risa/Asir, computer algebra system, developed by Fujitsu Laboratories. I thank him for his advice.

Prof. Kiyoko Nishizawa (Josai Univ.) informed me of the studies of non-monotone bifurcation, and game me pertinent advice. I would like to thank her for her direction.

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